# The Projective and Injective Tensor Products of Banach Spaces

**Hanuram Vikash**
**Supervisor: Dr. Amudhan Krishnaswamy-Usha**
**Indian Institute of Science Education and Research Bhopal**

## The Algebraic Tensor Product

The algebraic tensor product $X \otimes Y$ linearizes bilinear maps, in the sense that any bilinear map from $X \times Y$ to $Z$ factors through $X \otimes Y$ as a linear map.

$$\begin{array}{ccc} X \times Y & \xrightarrow{(x,y)\mapsto x\otimes y} & X \otimes Y \\ & \searrow^{A} & \downarrow L \\ & & Z \end{array}$$

One way to view $X \otimes Y$ is as a subspace of $B(X^\sharp \times Y^\sharp)$. In this treatment, an elementary tensor $x \otimes y$ is defined by $(x \otimes y)(A_1, A_2) = A_1(x)A_2(y)$, and $X \otimes Y$ is the linear span of all elementary tensors.

## If $X, Y$ are normed spaces, can $X \otimes Y$ also be endowed with a norm?

**Definition (Reasonable Crossnorm).** A norm, $\alpha$, on $X \otimes Y$ is a *reasonable crossnorm* if the following hold:

1. For every $x \in X$ and $y \in Y$, we have $\alpha(x \otimes y) \leq \|x\|\|y\|$, and
2. For every $A_1 \in X^*$ and $A_2 \in Y^*$, the linear functional $A_1 \otimes A_2$ is bounded, with $\|A_1 \otimes A_2\| \leq \|A_1\|\|A_2\|$.

## The Projective Tensor Product

**Definition (Projective Norm).** Let $u \in X \otimes Y$. The *projective norm* of $u$, denoted by $\pi(u)$, is:

$$\pi(u) = \inf \left\{ \sum_{i=1}^{n} \|x_i\|\|y_i\| \mid u = \sum_{i=1}^{n} x_i \otimes y_i \right\}. \tag{1}$$

The completion under this norm is the *projective tensor product* $X \hat{\otimes}_\pi Y$. The projective tensor product linearizes bounded bilinear mappings, similar to how the algebraic tensor product linearizes bilinear mappings.

## The Injective Tensor Product

**Definition (Injective Norm).** Let $u = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y$. The *injective norm of* $u$, denoted by $\varepsilon(u)$, is:

$$\varepsilon(u) = \sup \left\{ \left| \sum_{i=1}^{n} A_1(x_i)A_2(y_i) \right| \mid A_1 \in B_{X^*}, A_2 \in B_{Y^*} \right\}. \tag{2}$$

The injective norm of a tensor $u$ is the operator norm of the bilinear form associated with $u$. The closure (when viewed as elements of $B(X^\sharp \times Y^\sharp)$) under this norm is the *injective tensor product* $X \hat{\otimes}_\varepsilon Y$.

## We thus have, for every $u \in X \otimes Y$, $\pi(u) \geq \alpha(u) \geq \varepsilon(u)$ for any reasonable crossnorm $\alpha$.

## The Bochner Integral and $L_1(\mu, X)$

$(\Omega, \Sigma, \mu)$ is a measure space.

**Definition (Bochner Integral).** For a Banach space valued, $\mu$-measurable simple function $f = \sum_{i=1}^{n} \chi_{A_i} x_i$, the *Bochner integral over a measurable set* $E$, $\int_E f\,d\mu$ is defined as:

$$\int_E f\,d\mu := \sum_{i=1}^{n} \mu(A_i \cap E)x_i. \tag{3}$$

A $\mu$-measurable function $f$ is *Bochner integrable* if there is a sequence $f_n$ converging a.e to $f$ such that $\lim_{n\to\infty} \int_\Omega \|f - f_n\|d\mu = 0$. In this case, $\int_E f\,d\mu = \lim_{n\to\infty} \int_E f_n d\mu$.

**Theorem 1 (Lebesgue-Bochner space).** *The space of equivalence classes of Bochner integrable functions, denoted by $L_1(\mu, X)$, is a Banach space with the norm $\|f\|_1 = \int_\Omega \|f\|d\mu$.*

**Proposition 2.** *There is an isometric isomorphism $L_1(\mu)\hat{\otimes}_\pi X \simeq L_1(\mu, X)$.*

$$\begin{array}{ccc} L_1(\mu) \times X & \xrightarrow{B} & L_1(\mu)\hat{\otimes}_\pi X \\ & \searrow^{(f,x)\mapsto f\otimes x} & \uparrow J \\ & & L_1(\mu, X) \end{array}$$

## The Pettis integral extends the Bochner integral

Every Bochner integrable function is also Pettis integrable, and the integrals coincide. However, there exist Pettis integrable functions that are not Bochner integrable.

Consider the Lebesgue measure on [0,1] and the function $f : [0, 1] \to c_0$, defined as $f(t) = (n\chi_n(t))$ where $\chi_n$ is the characteristic function of the interval $\left(\frac{1}{n+1}, \frac{1}{n}\right]$. The Pettis integral $\int_\Omega f\,dm$ is $(\frac{1}{n+1}) \in c_0$, but $f$ is not Bochner integrable.

## The Pettis Integral and $\hat{P}_1(\mu, X)$

Let $(\Omega, \Sigma, \mu)$ be a finite measure space, and $f : \Omega \to X$ be a Banach space valued function.

**Definition (Weakly $\mu$-measurable Functions).** A function $f$ is *weakly $\mu$-measurable* if $A \circ f$ is $\mu$-measurable for every $A \in X^*$.

**Definition (Weakly Integrable Functions).** A weakly $\mu$-measurable function is *weakly integrable* if $A \circ f$ is integrable for every $A \in X^*$.

**Definition (Pettis Integral).** Given a weakly integrable function $f$, we define a bounded linear operator $S : X^* \to L_1(\mu)$ as $A \mapsto A \circ f$. The adjoint $S^* : L_\infty(\mu) \to X^{**}$ is such that, for any $g \in L_\infty(\mu)$, $\langle A, S^*g\rangle = \int_\Omega (g)(A \circ f)d\mu$. If the adjoint takes values in $X$ (when embedded into $X^{**}$), $f$ is *Pettis integrable* and the Pettis integral over $E$ is the image of $\chi_E$ under $S^*$.

**Definition (Pettis Norm).** Let $f$ be weakly integrable. The *Pettis norm*, denoted by $|f|_1$, is:

$$|f|_1 = \sup_{A \in B_{X^*}} \left\{ \int_\Omega |A \circ f|d\mu \right\} \tag{4}$$

$$= \sup_{g \in B_{L_\infty(\mu)}} \left\{ \left\| \int_\Omega g \cdot f \right\| \right\}. \tag{5}$$

**Theorem 3.** *The Pettis norm defines a norm on the space $P_1(\mu, X)$ of equivalence classes of $\mu$-measurable Pettis integrable functions. Its completion is denoted by $\hat{P}_1(\mu, X)$. Moreover, the $\mu$-measurable simple functions are dense in $\hat{P}_1(\mu, X)$.*

**Proposition 4.** *There is an isometric isomorphism $L_1(\mu)\hat{\otimes}_\varepsilon X \simeq \hat{P}_1(\mu, X)$.*

$$\begin{array}{ccc} L_1(\mu) \times X & \xrightarrow{B} & L_1(\mu)\hat{\otimes}_\varepsilon X \\ & \searrow^{(f,x)\mapsto f\otimes x} & \uparrow J \\ & & \hat{P}(\mu, X) \end{array}$$

## References

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